次の極限値を求めよ。（Ｎｏ．２）

(1) $\displaystyle\lim_{n\to\infty}\left[\log\{(n+1)(n+2)(n+3)\cdots(n+n)\}^{\frac{1}{n}}-\log n\right]$

$\displaystyle=\lim_{n\to\infty}\log\left\{\frac{(n+1)(n+2)(n+3)\cdots(n+n)}{n^n}\right\}^{\frac{1}{n}}$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{n+1}{n}\frac{n+2}{n}\frac{n+3}{n}\cdots\frac{n+n}{n}\right)$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\left(\log\frac{n+1}{n}+\log\frac{n+2}{n}+\log\frac{n+3}{n}+\cdots+\log\frac{n+n}{n}\right)$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)$

$\displaystyle=\int_0^1\log(1+x)dx=[(1+x)\log(1+x)]_0^1-\int_0^1dx=2\log 2-1$

(2) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\cos\left(\frac{k^2\pi}{2n^2}\right)$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\cos\left\{\frac{\pi}{2}\left(\frac{k}{n}\right)^2\right\}$

$\displaystyle=\int_0^1x\cos\frac{\pi}{2}x^2dx=\int_0^1\cos\frac{\pi}{2}t\left(\frac{1}{2}dt\right)=\frac{1}{2}\left[\frac{2}{\pi}\sin\frac{\pi}{2}t\right]_0^1=\frac{1}{2}\times\frac{2}{\pi}=\frac{1}{\pi}$

(3) $\displaystyle\lim_{n\to\infty}\int_0^1\sum_{k=0}^{n-1}x^{2n+k}dx$

$\displaystyle=\lim_{n\to\infty}\int_0^1(x^{2n}+x^{2n+1}+x^{2n+2}+\cdots+x^{3n-1})dx$

$\displaystyle=\lim_{n\to\infty}\left(\left[\frac{x^{2n+1}}{2n+1}+\frac{x^{2n+2}}{2n+2}+\frac{x^{2n+3}}{2n+3}+\cdots+\frac{x^{3n}}{3n}\right]_0^1\right)$

$\displaystyle=\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\frac{1}{2n+3}+\cdots+\frac{1}{3n}\right)$

$\displaystyle=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2n+k}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2+\frac{k}{n}}=\int_0^1\frac{1}{2+x}dx$

$\displaystyle=[\log|2+x|]_0^1=\log 3-\log 2=\log\frac{3}{2}$

(4) $\displaystyle\lim_{n\to\infty}\frac{(1+2+3+\cdots+n)^5}{(1+2^4+3^4+\cdots+n^4)^2}$

（分子）$\displaystyle=\left\{\frac{1}{2}n(n+1)\right\}^5$

（分母）$\displaystyle=\left[n^4\left\{\left(\frac{1}{n}\right)^4+\left(\frac{2}{n}\right)^4+\left(\frac{3}{n}\right)^4+\cdots+\left(\frac{n}{n}\right)^4\right\}\right]^2$

$\displaystyle=\left[n^5\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}\right]^2$

$\displaystyle=n^{10}\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2$

なので、

（与式）$\displaystyle=\lim_{n\to\infty}\frac{\frac{1}{32}\frac{(n+1)^5}{n^5}}{\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2}$

$\displaystyle=\lim_{n\to\infty}\frac{\frac{1}{32}\left(1+\frac{1}{n}\right)^5}{\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2}=\frac{\frac{1}{32}}{\left(\int_0^1x^4dx\right)^2}=\frac{1}{32}\frac{1}{\left(\left[\frac{1}{5}x^5\right]_0^1\right)^2}=\frac{25}{32}$

(5) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\log\left\{\frac{n}{n}\times\frac{n+2}{n}\times\frac{n+4}{n}\times\cdots\times\frac{n+2(n-1)}{n}\right\}$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{n+2k}{n}$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\left\{1+2\left(\frac{k}{n}\right)\right\}$

$\displaystyle=\int_0^1\log(1+2x)dx=\left[\frac{1}{2}(1+2x)\log(1+2x)\right]_0^1-\int_0^1dx=\frac{3}{2}\log 3-1$

(6) $\displaystyle\lim_{n\to\infty}\frac{1}{n^6}\sum_{k=n+1}^{2n}k^5$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=n+1}^{2n}\left(\frac{k}{n}\right)^5$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{n+k}{n}\right)^5$

$\displaystyle=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^5$

$\displaystyle=\int_0^1(1+x)^5dx=\left[\frac{1}{6}(1+x)^6\right]_0^1=\frac{64}{6}-\frac{1}{6}=\frac{63}{6}=\frac{21}{2}$